Unbounded Solution

In maximization LPP, if $C_j - Z_j > 0$($C_j - Z_j < 0$ for a maximization case) for a column not in the basis and all entries in this column are negative, then for determining key row, we have to calculate minimum ratio corresponding to each basic variable having negative or zero value in the denominator. Negative value in the denominator can not be considered, as it would indicate the entry of non-basic variable in the basis with a negative value (an infeasible solution will occur). A zero value in the denominator would result in ratio having a $+\infty$. This implies that the entering variable could be increased indefinitely with any of the current basic variables being removed from the basis. In general, an unbounded solution occurs due to wrong formulation of the problem within the constraint set, and thus needs reformulation.

Example: Solve the following LPP;

$$Max\ \ Z = 3x_1 + 5x_2$$

subject to the constraints

$$x_1 - 2x_2 \leq 6$$
$$x_1 \leq 10$$
$$x_2 \geq 1$$

and

$$x_1, x_2 \geq 0$$

Solution:

Adding slack variables S_1, S_2, surplus variable S_3 and artificial variable A_1 in the constraint set the LPP becomes;

$$Max\ Z = 3x_1 + 5x_2 + 0S_1 + 0S_2 + 0S_3 - MA_1$$

subject to the constraints

$$x_1 - 2x_2 + S_1 = 6$$
$$x_1 + S_2 = 10$$
$$x_2 - S_3 + A_1 = 1$$

and

$$x_1, x_2, S_1, S_2, S_3, A_1 \geq 0$$

The initial solution to this LPP is shown in Table 1

Table 1: Initial Solution

		$C_j \longrightarrow$	3	5	0	0	0	-M	
C_B	B	$b(=x_B)$	x_1	x_2	S_1	S_2	S_3	A_1	Min.Ratio
0	S_1	6	1	-2	1	0	0	0	-
0	S_2	10	1	0	0	1	0	0	-
-M	A_1	1	0	1	0	0	-1	1	$\frac{1}{1} = 1 \rightarrow$
$Z = -M$		Z_j	0	-M	0	0	M	-M	
		$C_j - Z_j$	3	5+M $\uparrow$	0	0	-M	0	

From Table 1, $C_2 - Z_2$ has largest positive value, thus variable x_2 enters the basis and A_1 leaves the basis. The new solution is shown in Table 2

Table 2: Improved Solution

		$C_j \longrightarrow$	3	5	0	0	0	-M
C_B	B	$b(=x_B)$	x_1	x_2	S_1	S_2	S_3	A_1
0	S_1	8	1	0	1	0	-2	2
0	S_2	10	1	0	0	1	0	0
5	x_2	1	0	1	0	0	-1	1
$Z = 5$		Z_j	0	5	0	0	-5	5
		$C_j - Z_j$	3	0	0	0	5	-M-5

From the Table 2, $C_1 - Z_1 = 3$ and $C_5 - Z_5 = 5$ entries are positive and $C_5 - Z_5 \geq C_1 - Z_1$. Therefore, variable S_3 should enter into the basis. Here it may be noted that coefficients in the 'S_3' column are all negative or zero. This indicates that S_3 cannot be entered into the basis. However, the value of S_3 can be increased infinitely without removing any one of the basic variables. Further, since S_3 is associated with x_1 in the third constraint, x_1 will also be increased infinitely because it can be expressed as $x_1 = 1 + S_3 - A_1$. Hence, the solution to the given LPP is unbounded.

Infeasible Solution

In the final simplex table, if atleast one of the artificial variable appears with a positive value, no feasible solution exists, because it is not possible to remove such an artificial variable from the basis using the simplex algorithm. When an infeasible solution exists, the LP Model should be reformulated. This may be because of the fact that the model is either improperly formulated or two or more of the constraints are incompatible.

Example:

$$Max\ \ Z = 6x_1 + 4x_2$$

subject to the constraints

$$x_1 + x_2 \leq 5$$
$$x_2 \geq 8$$

and

$$x_1, x_2 \geq 0$$

Solution:

By adding slack, surplus and artificial variables, the LPP becomes;

$$Max\ Z = 6x_1 + 4x_2 + 0S_1 + 0S_2 - MA_1$$

subject to the constraints

$$x_1 + x_2 + S_1 = 5$$
$$x_2 - S_2 + A_1 = 8$$

and

$$x_1, x_2, S_1, S_2, A_1 \geq 0$$

The initial solution to this LPP is shown in Table 1

Table 1: Initial Solution

C_B	B	$C_j \longrightarrow$ $b(=x_B)$	6 x_1	4 x_2	0 S_1	0 S_2	-M A_1	Min.Ratio
0	S_1	5	1	1	1	0	0	1 = 5 →
-M	A_1	8	0	1	0	-1	1	1 = 8
$Z = -8M$		Z_j	0	-M	0	M	-M	
		$C_j - Z_j$	6	4+M ↑	0	-M	0	

Variable x_2 enters the basis and S_1 leaves the basis. The new solution is shown in Table 2

Table 2

C_B	B	$C_j \longrightarrow$ $b(=x_B)$	6 x_1	4 x_2	0 S_1	0 S_2	-M A_1
4	x_2	5	1	1	1	0	0
-M	A_1	3	-1	0	-1	-1	1
$Z = 20 - 3M$		Z_j	4+M	4	4+M	M	-M
		$C_j - Z_j$	2-M	0	-4-M	-M	0

Since all $C_j - Z_j \leq 0$, the solution shown in Table 4.28 is optimal. But this solution is not feasible for the given problem since it has $x_1 = 0$ and $x_2 = 5$ (recall that in the second constraint $x_2 \geq 8$). The fact that artificial variable $A_1 = 3$ is in the solution also indicates that the final solution violates the second constraint.